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★**Quantization on nilpotent Lie groups.**

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As the authors assert in the Introduction, the main topic of this prize-winning monograph is the development of a pseudo-differential calculus on homogeneous Lie groups—the nilpotent Lie groups equipped with a family of dilations compatible with the group structure.

The principal tool in this study is the process called the quantization on a group. The quantization allows one to describe an operator acting on the space of functions on a group by means of the operator symbol.

For a locally compact unimodular group G, denote by $\widehat{G}$ the unitary dual of G, namely the space of equivalence classes of unitary irreducible representations of G. Let μ be the Plancherel measure on $\widehat{G}$. The symbol of a linear operator T acting on a certain vector space of functions on G is a map which assigns to $(x,\pi) \in G \times \widehat{G}$ an operator $\sigma(x,\pi)$ defined on the carrier space H_π of π in such a way that

$$T\phi(x) = \int_{\widehat{G}} \mathrm{Tr}(\pi(x)\sigma(x,\pi)\widehat{\phi}(\pi))d\mu(\pi).$$

In order to carry out the quantization on a group G, it is necessary to determine a class of symbols for which the formula above makes sense, at least for a suitable space of functions ϕ, as well as to find a form of calculating the symbol σ for a given operator T.

Clearly, the quantization in this sense is a very general, complicated and rather enigmatic program, which involves many problems of classical harmonic analysis.

The simplest non-trivial example of a homogeneous Lie group is the Heisenberg group, whose connections with quantum mechanics, optics, signal analysis, and thermodynamics show that the quantization on nilpotent Lie groups is a subject of interest for numerous branches of science.

It was a good decision by the authors to include (in Chapter 2) the case of the quantization on a compact Lie group, although from a technical point of view this case is quite different from the case of a nilpotent group. The existence of a finite invariant measure on the group, the finite-dimensionality of the irreducible representations, and the existence of the elliptic Laplace operator make the quantization and developing of the theory of pseudo-differential operators for compact Lie groups much simpler.

Although the nilpotent case requires different, much more complicated methods, by studying the compact group case first we can appreciate the role played by the quantization of the Laplace operator in the study of pseudo-differential operators.

It is thus natural that, after describing in Chapter 3 the structure of homogeneous Lie groups (as well as of graded and stratified Lie groups), the authors concentrate their attention on the study of invariant differential operators on homogeneous Lie groups, which in some sense take the part of the Laplace operator.

Chapter 4 is dedicated to the study of Rockland operators and Sobolev spaces on homogeneous Lie groups, and provides a foundation for the realization of the quantization program. The dilation structure of a homogeneous Lie group permits one to distinguish the classes of homogeneous operators and of Rockland operators. The Rockland operators are the left-invariant, homogeneous and hypoelliptic differential operators. The point is that for a left-invariant, homogeneous differential operator $\mathcal{R}$, the hypoellipticity is equivalent to the property called the Rockland condition, which states that for every $\mathrm{id} \neq \pi \in \widehat{G}$, the operator $d\pi(\mathcal{R})$ is injective on its domain H_π^∞. Obviously, this condition is fundamental to the study of the quantization of $\mathcal{R}$.

After investigating the relation between the hypoellipticity and the Rockland condition, the authors develop the functional calculus of formally self-adjoint Rockland operators on graded Lie groups, following this with the analysis of the associated heat semigroup.

The remaining part of Chapter 4 is the first systematic presentation of the fractional powers of Rockland operators and the homogeneous and inhomogeneous Sobolev spaces associated with a positive Rockland operator on a graded Lie group. Although the Sobolev spaces are constructed using a particular positive Rockland operator $\mathcal{R}$, the result of the construction is independent of the choice of $\mathcal{R}$.

Besides the description of the Sobolev spaces in $L^p(G)$ for $1 < p < \infty$ and their duals, the chapter presents the interpolation theorems between these spaces, the definition and properties of Sobolev imbeddings, and the characterization of the action of differential operators on these spaces.

The chapter ends with the study of kernels of the multiplier operators $\phi(\mathcal{R})$ for ϕ in the Schwartz space on G and a Rockland operator $\mathcal{R}$. Using Hulanicki's theorem (the complete proof of it is included), the authors prove that the kernel of $\phi(\mathcal{R})$ belongs to the Schwartz space on G.

The principal aim of the book, developing the theory of pseudo-differential operators on graded Lie groups, is accomplished in Chapter 5. The task of determining a class of symbols and the corresponding operators is performed using a positive Rockland operator, but again it appears that the effect of the construction does not depend on the chosen operator. The most arduous part of the job is the construction of spaces which serve as domains of the symbols and the authors deserve credit for making this section interesting.

The space of operators obtained in this way is a $*$-algebra. The elliptic and hypoelliptic elements of this algebra admit the parametrix construction and the symbol of the parametrix can be obtained from the symbol of the operator in question.

The study of the kernels of these operators shows that the operators are Calderón-Zygmund. An analogue of the Calderón-Vaillancourt theorem is proved.

The last chapter provides an application of the theory to pseudo-differential operators on the Heisenberg group, which is an example of a stratified (hence graded) Lie group. The approach based on the method presented in Chapter 5 is different from the traditional approach in the literature. It is proved that the analysis of pseudo-differential operators on the Heisenberg group can be reduced to considering scalar-valued symbols parametrized by the elements of the Heisenberg group and a positive parameter λ. The so-called λ-symbols are of Shubin type.

This book has been awarded the 2014 Ferran Sunyer i Balaguer Prize and deserves to be recognized for a number of reasons.

Using just the classical methods of functional and harmonic analysis, the authors create an advanced theory and present it employing intelligently chosen and relatively simple notation.

The extensive Introduction is a fascinating essay about the history, trends, and current state of knowledge about functional analysis on nilpotent groups and the theory of differential and pseudo-differential operators. This exposition demonstrates impressive erudition of the authors and their familiarity with the extensive related literature, as well as with the contributions of particular researchers in the development of the theory. Moreover, the structure of the book is described here very clearly. All chapters are provided with additional explications, so that the reader is always well informed about the aims of a given section.

An important part of the monograph is the authors' own contribution to the theory. The original and creative manner of presentation of the material known from other publications constitutes additional merit.

It is really surprising that in spite of its great length and complicated subject, this book is very accessible. *Antoni Wawrzyńczyk*