Hardy inequalities on homogeneous groups

(100 years of Hardy inequalities)

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Preface

The subject of Hardy inequalities has now been a fascinating subject of continuous research by numerous mathematicians for exactly[1] one century, 1918-2018.

It appears to have been inspired by D. Hilbert's investigations in the theory of integral equations where he came across a beautiful fact that the series

$$\sum_{m,n=1}^{\infty} \frac{a_m a_n}{m+n}$$

with positive entires $a_n \geq 0$ is convergent whenever $\sum_{m=1}^{\infty} a_m^2$ is convergent. In a few years period at least four different proofs of this fact have been published: the original proof of Hilbert given by his doctoral student H. Weyl in 1908 in his Inaugural-Dissertation [Wey08a, Page 83] also appearing in [Wey08b], a proof by F. Wiener [Wie10] in 1910, and two proofs by I. Schur [Sch11] in 1911. All these proofs including the Wiener's proof in the paper bearing the title "Elementarer Beweis eines Reihensatzes von Herrn Hilbert" were still not considered elementary enough by G. H. Hardy, so he came up in 1918 with yet another proof in [Har19] which seemed to him "to lack nothing in simplicity". In fact, there, he derived Hilbert's theorem as a simple 3-line corollary to the following statement: if the series $\sum_{m=1}^{\infty} a_m^2$ is convergent and we set $A_n := a_1 + \cdots + a_n$, then also the series

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n} \right)^2$$

is convergent. Thus, this moment could be considered as the birth of what is now known as Hardy's inequalities, although Hardy himself reservedly commented on his theorem with "it seems to be of some interest in itself".

After G. H. Hardy communicated his proof to Marcel Riesz, at once Riesz came up with another argument leading to the following generalisation of Hardy's

[1] The original inequality was published by G. H. Hardy in "Notes on some points in the integral calculus (51)", Messenger of Mathematics, 48 (1918), pp 107-112, see the note in [Har20, Footnote 4] for a historic remark.

result: if $\varkappa > 1$ and $\sum_{m=1}^{\infty} a_m^{\varkappa}$ is convergent, then also the series

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n}\right)^{\varkappa}$$

is convergent. Thus, this can be also regarded as the birth of what is now known as L^p-Hardy's inequalities (but should be probably then called Hardy-Riesz inequalities). The proof of Riesz and the historical account of this matter was then published as a short note[2] by Hardy in [Har20]. Hardy also gave the exact value of the best constant in the inequality, together with its extension to the integral formulation in the form of

$$\int_a^{\infty} \left(\frac{\int_a^x f(t)dt}{x}\right)^{\varkappa} dx \leq \left(\frac{\varkappa}{\varkappa - 1}\right)^{\varkappa} \int_a^{\infty} f^{\varkappa} dx,$$

where a and f are positive. Interestingly, Hardy called his own proof for the best constant "unnecessarily complicated", so in [Har20] he gave another simpler proof that was "sent to him by Prof. Schur by letter".

Over the last 100 years the subject of Hardy inequalities and related analysis has been a topic of intensive research: currently MathSciNet lists more than 800 papers containing words 'Hardy inequality' in the title, and almost 3500 papers containing words 'Hardy inequality' in the abstract or in the review. In view of this wealth of information *we apologise for the inevitability of missing to mention many important contributions to the subject.*

Nevertheless, the Hardy inequalities with many references have been already presented in many monographs and reviews; here we can mention excellent presentations by Opic and Kufner [OK90] in 1990, Davies [Dav99] in 1999, Kufner and Persson [KP03] (and with Samko [KPS17]), Edmunds and Evans [EE04] in 2004, part of Mazya's books [Maz85, Maz11], Ghoussoub and Moradifam [GM13] in 2013, and Balinsky, Evans and Lewis [BEL15] in 2015, as well as books on different areas related to Hardy inequalities: Hardy inequalities on time scales [AOS16], Hardy inequalities with general kernels [KHPP13], weighted Hardy inequalities [KP03], Hardy inequalities and sequence spaces [GE98]. The history and prehistory of Hardy inequalities were discussed in [KMP07] and in [KMP06], respectively, also with 'what should have happened if Hardy had discovered this' considerations [PS12].

However, all of these presentations are largely confined to the Euclidean part of the available wealth of information on this subject.

At the same time there is another layer of intensive research over the years related to Hardy inequalities in sub-elliptic settings motivated by their applications

[2] It seems Hardy liked publishing such notes as, according to MathSciNet, 51 of his papers start with words "A note on...", together with papers titled "Additional note on..." or "A further note on..."

to problems involving sub-Laplacians. This is complemented by the more general anisotropic versions of the theory.

In this direction, the sub-elliptic ideas of the analysis on the Heisenberg group, significantly advanced by Folland and Stein in [FS74], were subsequently consistently developed by Folland [Fol75] leading to the foundations for analysis on stratified groups (or homogeneous Carnot groups). Furthermore, in their fundamental book [FS82] titled "**Hardy spaces on homogeneous groups**", Folland and Stein laid down foundations for the 'anisotropic' analysis on general homogeneous groups, i.e. Lie groups equipped with a compatible family of dilations. Such groups are necessarily nilpotent, and the realm of homogeneous groups almost exhausts the whole class of nilpotent Lie groups including the classes of stratified, and more generally, graded groups. Happily, *the title of our monograph pays tribute to G. H. Hardy as well as to Folland and Stein's book.*

Among many, one of the motivations behind doing analysis on homogeneous groups is the "distillation of ideas and results of harmonic analysis depending only on the group and dilation structures".

The place where Hardy inequalities and homogeneous groups meet is a beautiful area of mathematics which was not consistently treated in the book form. We took it as an incentive to write this monograph to collect and deepen the understanding of Hardy inequalities and closely related topics from the point of view of Folland and Stein's homogeneous groups. While we describe the general theory of Hardy, Rellich, Caffarelli-Kohn-Nirenberg, Sobolev, and other inequalities in the setting of general homogeneous groups, a particular attention is paid to the special class of stratified groups. In this setting the theory of Hardy inequalities becomes intricately intertwined with the properties of sub-Laplacians and subelliptic partial differential equations.

These topics constitute the core of this book with the material complemented with additional closely related topics such as uncertainty principles, function spaces on homogeneous groups, the potential theory for stratified groups, and the potential theory for general Hörmander's sums of squares and their fundamental solutions.

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Contents