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★**Pseudo-differential operators and symmetries.**
Background analysis and advanced topics.
Pseudo-Differential Operators. Theory and Applications, 2.
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The present book is devoted to the theory of pseudodifferential operators on spaces with symmetries, namely the Euclidean space $\mathbb{R}^n$, the torus $\mathbb{T}^n$, compact Lie groups and homogeneous spaces.

Pseudodifferential operators in $\mathbb{R}^n$ were introduced in 1965 by J. J. Kohn, L. Nirenberg and, independently, by L. Hörmander, after previous work by S. G. Mikhlin, A. Calderón, A. S. Zygmund and P. Lax. The invariance of the operator classes with respect to smooth changes of variables allows one to transfer the definition on smooth manifolds by local coordinates. One of the first successes of the theory of pseudodifferential operators on manifolds was the celebrated Atiyah-Singer Index Theorem. Since then, several generalizations have been considered, to manifolds with boundary and corners, with cone or edge singularities, etc.

However, this local approach is plagued by local coordinates issues. For example, while the principal symbol of a classical pseudodifferential operator has an invariant meaning, this is no longer the case for the full symbol. Hence many problems in $\mathbb{R}^n$ where the full symbol plays an essential role cannot be transferred directly to manifolds.

The principle that inspired this book is that a *global* approach can still be carried out for manifolds that enjoy some symmetry. The main example (after the Euclidean space) is the n-dimensional torus $\mathbb{T}^n$. Indeed, pseudodifferential operators on $\mathbb{T}^n$ can be defined formally by

$$Af(x)=\sum_{\xi\in\mathbb{Z}^n} e^{2\pi i x\xi}\sigma_A(x,\xi)\widehat{f}(\xi),\quad x\in\mathbb{T}^n,$$

where $\widehat{f}(\xi)$, $\xi\in\mathbb{Z}^n$, are the Fourier coefficients of f. The symbol $\sigma_A(x,\xi)$, $x\in\mathbb{T}^n$, $\xi\in\mathbb{Z}^n$, satisfies the estimates

$$\left|\Delta_\xi^\alpha\partial_x^\beta\sigma_A(x,\xi)\right|\le C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|},\quad \forall\alpha,\beta\in\mathbb{N}^n,$$

for some $m\in\mathbb{R}$, where Δ^α is a finite-difference operator. These classes of operators were first introduced by M. S. Agranovich [Funktsional. Anal. i Prilozhen. **13** (1979), no. 4, 54–56; MR0554412], and were proved to be the same as Hörmander's classes of pseudo-differential operators in $\mathbb{R}^n$ with symbol $\mathbb{Z}^n$-periodic in the first variable.

The theory of pseudo-differential operators on the torus is quite similar to that in the Euclidean spaces, due to the commutative nature of the underlying space. The main purpose of the present book is to extend this global approach (i.e. without referring to local coordinates) to the general case of a compact Lie group G. The starting point is given by the representation theory of compact Lie groups, and the notion of Fourier transform.

More precisely, consider a strongly continuous irreducible unitary representation $\xi\colon G\to\mathcal{U}(\mathcal{H}_\xi)$ (the representation space $\mathcal{H}_\xi$ is finite-dimensional, since G is compact).

Then the Fourier coefficient $\widehat{f}(\xi) \in \mathrm{End}(\mathcal{H}_\xi)$, $f \in L^1(G)$, is defined as standardly by

$$(\widehat{f}(\xi)u, v)_{\mathcal{H}_\xi} = \int f(x)(\xi(x)^* u, v)_{\mathcal{H}_\xi} dx, \quad u, v \in \mathcal{H}_\xi.$$

Pseudodifferential operators on G are then defined as operators of the form

$$Af(x) = \sum_{[\xi] \in \widehat{G}} \dim(\xi) \mathrm{Tr}\big(\xi(x)\sigma_A(x,\xi)\widehat{f}(\xi)\big), \quad x \in G,$$

where $\dim(\xi) := \dim \mathcal{H}_\xi$, and $\sigma_A(x,\xi) \in \mathrm{End}(\mathcal{H}_\xi)$ is the symbol of A. Due to the presence of the trace, $Af(x)$ does not depend on the choice of the representative ξ of the residue class $[\xi]$ in the unitary dual $\widehat{G}$.

Hence, contrary to the local approach, we have here a notion of *full symbol* available. In fact, one can also define symbol classes in a way similar to the case of operators on the torus, where the weight $\langle \xi \rangle$ is now defined in terms of the eigenvalues of the Laplace operator on G.

Let us briefly review the contents of the book.

In Part I, "Foundations of analysis", basic results from real analysis are presented, with complete proofs. These include topics from functional analysis, measure theory and integration, and Banach algebras. This first part, quite extensive, makes the book self-contained; also, it could be used independently for a graduate course on these topics. In fact, several exercises are provided as well.

Part II concerns pseudodifferential operators on $\mathbb{R}^n$ and on $\mathbb{T}^n$. On the torus, even the more general classes of type (ρ, δ) are considered in detail, for no comprehensive treatment of them seems to be available in the literature. A characterization in terms of commutators is given also.

Part III provides the needed background from representation theory of compact Lie groups, and is also suitable for a first course on that topic.

Finally, Part IV deals with the theory of pseudodifferential operators on general compact Lie groups. The case of the group $\mathrm{SU}(2)$ is investigated in great detail. The study of homogeneous spaces and issues related to liftings are considered in the last chapter. Here the main example is given by the sphere $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$. Part of these results were contained in the Ph.D. thesis of the second author ["Pseudodifferential calculus on compact Lie groups and homogeneous spaces", Helsinki Univ. Technol., Espoo, 2001].

Summing up, this welcome book is nicely written and looks very appealing for researchers in the fields of Euclidean and abstract harmonic analysis and pseudodifferential operators, as well as for any Ph.D. student interested in those topics. *Fabio Nicola*