This monograph is devoted to the development of the theory of pseudo-differential operators on spaces with symmetries. Such spaces are the Euclidean space $\mathbb{R}^n$, the $n$-torus $\mathbb{T}^n$, compact Lie groups, and compact homogeneous spaces. The book consists of several parts. One of our aims has been not only to present new results on pseudo-differential operators but also to show parallels between different approaches to pseudo-differential operators on different spaces. Moreover, we tried to present the material in a self-contained way to make it accessible for readers approaching the material for the first time.

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### Quantization of operators

Let $\sigma_A$ be the symbol of a continuous linear operator $A : C^\infty(G) \to C^\infty(G)$. Then

$$A f(x) = \sum_{[\xi] \in \hat{G}} d\xi \text{Tr} \left( \xi(x) \sigma_A(x, \xi) \hat{f}(\xi) \right),$$

for every $f \in C^\infty(G)$ and $x \in G$. Conversely, we have

$$\sigma_A(x, \xi) = (\xi(x))^* A(x) \in C^{d_\xi \times d_\xi}_{C^\infty(G)}.$$

### Fourier Series on compact Lie groups

If $\xi : G \to U(d_\xi)$ is a unitary matrix representation of a compact Lie group $G$, then

$$\hat{f}(\xi) = \int_G f(x) \xi(x)^* dx \in C^{d_\xi \times d_\xi}$$

has matrix elements

$$\hat{f}(\xi)_{mn} = \int_G f(x) \xi(x)_{mn} dx \in C, \ 1 \leq m, n \leq d_\xi.$$

If $f \in L^2(G)$ then

$$\hat{f}(\xi)_{mn} = (f, \xi_{mn})_{L^2(G)},$$

and by the Peter–Weyl Theorem we have

$$f(x) = \sum_{[\xi] \in \hat{G}} d\xi \text{Tr} \left( \xi(x) \hat{f}(\xi) \right),$$

for almost every $x \in G$, where the summation is understood so that for each class $[\xi] \in \hat{G}$ we pick just (any) one representative $\xi \in [\xi]$. 

### Information

QR code for downloading the book
michael.ruzhansky@ugent.be
ville.turunen@aalto.fi
Please visit:
ruzhansky.org and analysis-pde.org